7.3 Joint space schemes

In this section we consider methods of path generation in which the path shapes (in space and in time) are described in terms of functions of joint angles.

Each path point is usually specified in terms of a desired position and orientation of the tool frame, $\{T\}$, relative to the station frame, $\{S\}$. Each of these via points is "converted" into a set of desired joint angles by application of the inverse kinematics. Then a smooth function is found for each of the n joints which pass through the via points and end at the goal point. The time required for each segment is the same for each joint so that all joints will reach the via point at the same time, thus resulting in the desired Cartesian position of $\{T\}$ at each via point. Other than specifying the same duration for each joint, the determination of the desired joint angle function for a particular joint does not depend on the functions for the other joints.

Hence, joint space schemes achieve the desired position and orientation at the via points. In between via points the shape of the path, while rather simple in joint space, is complex if described in Cartesian space. Joint space schemes are usually the easiest to compute, and, because we make no continuous correspondence between joint space and Cartesian space, there is essentially no problem with singularities of the mechanism.

Cubic polynomials

Consider the problem of moving the tool from its initial position to a goal position in a certain amount of time. Using the inverse kinematics the set of joint angles that correspond to the goal position and orientation can be calculated. The initial position of the manipulator is also known in the form of a set of joint angles. What is required is a function for each joint whose value at t_0 is the initial position of the joint, and whose value at t_f is the desired goal position of that joint. As shown in Fig. 7.2, there are many smooth functions, $\theta(t)$, which might be used to interpolate the joint value.

In making a single smooth motion, at least four constraints on $\theta(t)$ are evident. Two constraints on the function's value come from the selection of initial and final values:

$$\begin{aligned} \theta(0) &= \theta_0, \\ \theta(t_f) &= \theta_f. \end{aligned} \tag{7.1}$$

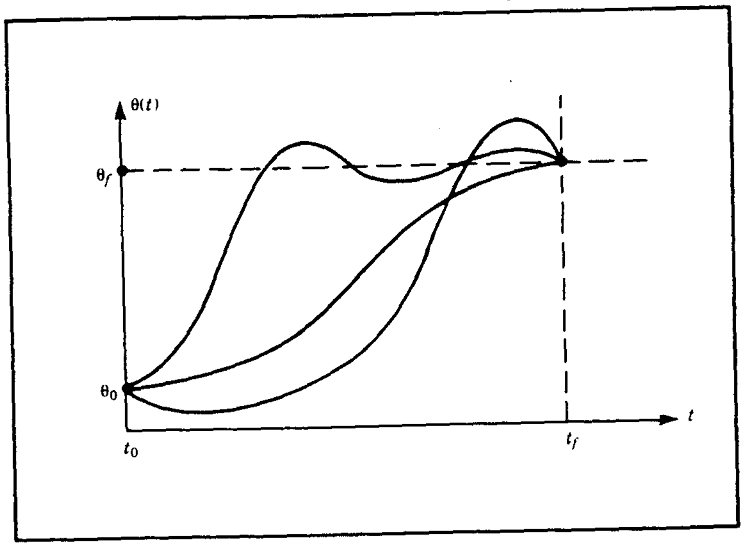

FIGURE 7.2 Several possible path shapes for a single joint.

An additional two constraints are that the function is continuous in velocity, which in this case means the the initial and final velocity are zero:

$$\begin{aligned} \dot{\theta}(0) &= 0, \\ \dot{\theta}(t_f) &= 0. \end{aligned} \tag{7.2}$$

These four constraints can be satisfied by a polynomial of at least third degree. Since a cubic polynomial has four coefficients, it can be made to satisfy the four constraints given by (7.1) and (7.2). These constraints uniquely specify a particular cubic. A cubic has the form

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \tag{7.3}$$

and so the joint velocity and acceleration along this path are clearly

$$\begin{aligned} \dot{\theta}(t) &= a_1 + 2a_2 t + 3a_3 t^2, \\ \ddot{\theta}(t) &= 2a_2 + 6a_3 t. \end{aligned} \tag{7.4}$$

Combining (7.3) and (7.4) with the four desired constraints yields four equations in four unknowns:

$$\begin{aligned} \theta_0 &= a_0, \\ \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3, \\ 0 &= a_1, \\ 0 &= a_1 + 2a_2 t_f + 3a_3 t_f^2. \end{aligned} \tag{7.5}$$

Solving these equations for the a_i we obtain

$$\begin{aligned} a_0 &= \theta_0, \\ a_1 &= 0, \\ a_2 &= \frac{3}{t_f^2}(\theta_f - \theta_0), \\ a_3 &= -\frac{2}{t_f^3}(\theta_f - \theta_0). \end{aligned} \tag{7.6}$$

Using (7.6) we can calculate the cubic polynomial that connects any initial joint angle position with any desired final position. This solution is for the case when the joint starts and finishes at zero velocity.

EXAMPLE 7.1

A single-link robot with a rotary joint is motionless at $\theta = 15$ degrees. It is desired to move the joint in a smooth manner to $\theta = 75$ degrees in 3 seconds. Find the coefficients of a cubic which accomplishes this motion and brings the manipulator to rest at the goal. Plot the position, velocity, and acceleration of the joint as a function of time.

Plugging into (7.6) we find

$$\begin{aligned} a_0 &= 15.0, \\ a_1 &= 0.0, \\ a_2 &= 20.0, \\ a_3 &= -4.44. \end{aligned} \tag{7.7}$$

Using (7.3) and 7.4) we obtain

$$\begin{aligned} \theta(t) &= 15.0 + 20.0t^2 - 4.44t^3, \\ \dot{\theta}(t) &= 40.0t - 13.33t^2, \\ \ddot{\theta}(t) &= 40.0 - 26.66t. \end{aligned} \tag{7.8}$$

Figure 7.3 shows the position, velocity, and acceleration functions for this motion sampled at 40 Hz. Note that the velocity profile for any cubic function is a parabola, and the acceleration profile is linear. ■

Cubic polynomials for a path with via points

So far we have considered motions described by a desired duration and a final goal point. In general, we wish to allow paths to be specified which include intermediate via points. If the manipulator is to come to rest at each via point, then we can use the cubic solution of Section 7.3.

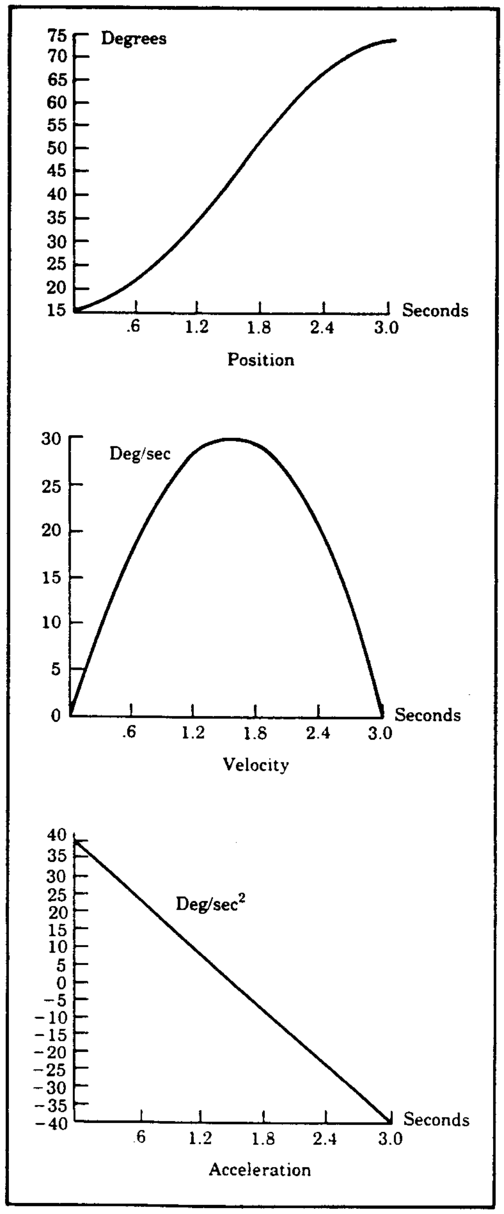

FIGURE 7.3 Position, velocity, and acceleration profiles for a single cubic segment which starts and ends at rest.

Usually, we wish to be able to pass through a via point without stopping, and so we need to generalize the way in which we fit cubics to the path constraints.

As in the case of a single goal point, each via point is usually specified in terms of a desired position and orientation of the tool frame relative to the station frame. Each of these via points is "converted" into a set of desired joint angles by application of the inverse kinematics. We then consider the problem of computing cubics which connect the via point values for each joint together in a smooth way.

If desired velocities of the joints at the via points are known, then we can determine cubic polynomials as before, but now the velocity constraints at each end are not zero, but rather, some known velocity. The constraints of (7.3) become

$$\begin{aligned} \dot{\theta}(0) &= \dot{\theta}_0, \\ \dot{\theta}(t_f) &= \dot{\theta}_f. \end{aligned} \tag{7.9}$$

The four equations describing this general cubic are

$$\begin{aligned} \theta_0 &= a_0, \\ \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3, \\ \dot{\theta}_0 &= a_1, \\ \dot{\theta}_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2. \end{aligned} \tag{7.10}$$

Solving these equations for the a_i we obtain

$$\begin{aligned} a_0 &= \theta_0, \\ a_1 &= \dot{\theta}_0, \\ a_2 &= \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f, \\ a_3 &= -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{1}{t_f^2}(\dot{\theta}_f + \dot{\theta}_0). \end{aligned} \tag{7.11}$$

Using (7.11) we can calculate the cubic polynomial that connects any initial and final positions with any initial and final velocities.

If we have the desired joint velocities at each via point, then we simply apply (7.11) to each segment to find the required cubics. There are several ways in which desired velocity at the via points might be specified.

1. The user specifies the desired velocity at each via point in terms of a Cartesian linear and angular velocity of the tool frame at that instant.
2. The system automatically chooses the velocities at the via points by applying a suitable heuristic in either Cartesian space or joint space.
3. The system automatically chooses the velocities at the via points in such a way as to cause the acceleration at the via points to be continuous.

In the first option. Cartesian desired velocities at the via points are "mapped" to desired joint rates using the inverse Jacobian of the manipulator evaluated at the via point. If the manipulator is at a singular point at a particular via point. then the user is not free to assign an arbitrary velocity at this point. While it is a useful capability of a path generation scheme to be able to meet a desired velocity which the user specifies, it would be a burden to require that the user always make these specifications. Therefore, a convenient system should include either option 2 or 3 (or both).

In option 2, the system automatically chooses reasonable intermediate velocities using some kind of heuristic. Consider the path specified by the via points shown for some joint, θ, in Fig. 7.4.

In Fig. 7.4 we have made a reasonable choice of joint velocities at the via points, as indicated with small line segments representing tangents to the curve at each via point. This choice is the result of applying a conceptually and computationally simple heuristic. Imagine the via points connected with straight line segments—if the slope of these lines changes sign at the via point, choose zero velocity, if the slope of these

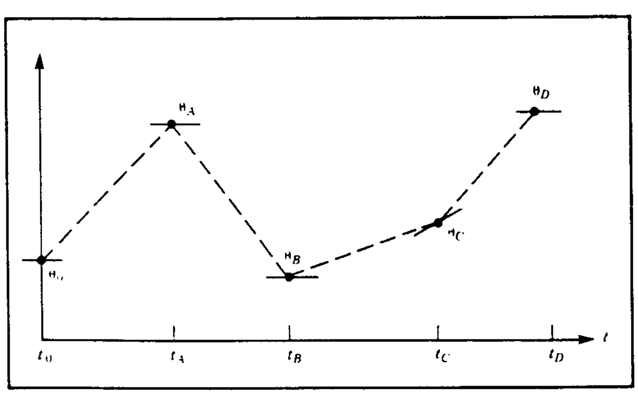

FIGURE 7.4 Via points with desired velocities at the points indicated by tangents.

lines does not change sign, choose the average of the two slopes as the via velocity. In this way, from specification of the desired via points alone, the system can choose the velocities at each point.

In option 3, the system chooses velocities such that acceleration is continuous at the via point. To do this, a new splining solution is needed. In this kind of spline, we replace the (two) velocity constraints at the connection of two cubics with the (two) constraints that a) velocity be continuous and b) acceleration be continuous.

EXAMPLE 7.2

Solve for the coefficients of two cubics which are connected in a two-segment spline with continuous acceleration at the intermediate via point. The initial angle is θ_0, the via point is θ_v, and the goal point is θ_g.

The first cubic is

$$\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3, \tag{7.12}$$

and the second is

$$\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3. \tag{7.13}$$

Each cubic will be evaluated over an interval starting at $t = 0$ and ending at $t = t_{fi}$, where $i = 1$ or $i = 2$.

The constraints we wish to enforce are

$$\begin{aligned}
\theta_0 &= a_{10}, \\
\theta_v &= a_{10} + a_{11}t_{f1} + a_{12}t_{f1}^2 + a_{13}t_{f1}^3, \\
\theta_v &= a_{20}, \\
\theta_g &= a_{20} + a_{21}t_{f2} + a_{22}t_{f2}^2 + a_{23}t_{f2}^3, \\
0 &= a_{11}, \\
0 &= a_{21} + 2a_{22}t_{f2} + 3a_{23}t_{f2}^2, \\
a_{11} + 2a_{12}t_{f1} + 3a_{13}t_{f1}^2 &= a_{21}, \\
2a_{12} + 6a_{13}t_{f1} &= 2a_{22}.
\end{aligned} \tag{7.14}$$