Moreover, applying (2) to the unit vectors **i**, **j**, **k**, we have

(9) $$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

Hence, if we write

$$\mathbf{A}\cdot\mathbf{B} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot(b_1\mathbf{i} + b_1\mathbf{j} + b_3\mathbf{k})$$

and use the fact that dot multiplication is distributive over addition [Eq. (3)] to expand and simplify, we obtain the important and now familiar result

(10) $$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$$

In particular, taking $\mathbf{B} = \mathbf{A}$, we have

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 = A^2 = a_1{}^2 + a_2{}^2 + a_3{}^2$$

or

(11) $$|\mathbf{A}| = A = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$$

On the other hand, if we write $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$ and then solve for $\cos\theta$, using (10) and (11), we obtain the useful formula

(12) $$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}\,\sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2}}$$

a result familiar from analytic geometry, where the a's and b's were introduced not as the components of two vectors but as the direction numbers of two straight lines.

For the cross products of the unit vectors **i**, **j**, **k** we find at once

(13) $$\begin{aligned} \mathbf{i}\times\mathbf{i} &= \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}\\ \mathbf{i}\times\mathbf{j} &= -\mathbf{j}\times\mathbf{i} = \mathbf{k}\\ \mathbf{j}\times\mathbf{k} &= -\mathbf{k}\times\mathbf{j} = \mathbf{i}\\ \mathbf{k}\times\mathbf{i} &= -\mathbf{i}\times\mathbf{k} = \mathbf{j} \end{aligned}$$

Hence, using (13) and the fact that cross multiplication is distributive over addition, we obtain for

$$\mathbf{A}\times\mathbf{B} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

the expression

(14) $$\mathbf{A}\times\mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

which is precisely the expanded form of the determinant

(15) $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3 \end{vmatrix}$$

The anticommutative character of vector multiplication thus corresponds to the fact that interchanging two rows of a determinant changes the sign of the determinant.

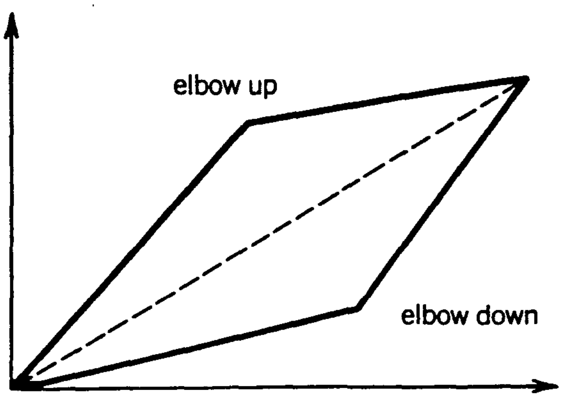

FIGURE 1-27
Multiple inverse kinematic solutions.

Consider the diagram of Figure 1-28. Using the **Law of Cosines** we see that the angle θ_2 is given by

$$\cos\theta_2 = \frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2} := D \tag{1.5.5}$$

We could now determine θ_2 as

$$\theta_2 = \cos^{-1}(D) \tag{1.5.6}$$

However, a better way to find θ_2 is to notice that if $\cos(\theta_2)$ is given by (1.5.5) then $\sin(\theta_2)$ is given as

$$\sin(\theta_2) = \pm\sqrt{1-D^2} \tag{1.5.7}$$

and, hence, θ_2 can be found by

$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-D^2}}{D} \tag{1.5.8}$$

The advantage of this latter approach is that both the elbow-up and elbow-down solutions are recovered by choosing the positive and negative signs in (1.5.8), respectively.

It is left as an exercise (Problem 1-19) to show that θ_1 is now given as

$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{a_2\sin\theta_2}{a_1+a_2\cos\theta_2}\right) \tag{1.5.9}$$

Notice that the angle θ_1 depends on θ_2. This makes sense physically since we would expect to require a different value for θ_1 depending on which solution is chosen for θ_2.

1.5.3 PROBLEM 3: VELOCITY KINEMATICS

In order to follow a contour at constant velocity, or at any prescribed velocity, we must know the relationship between the velocity of the tool and the joint velocities. In this case we can differentiate Equations 1.5.1 and 1.5.2 to obtain

$$\dot{x} = -a_1\sin\theta_1\cdot\dot{\theta}_1 - a_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \tag{1.5.10}$$

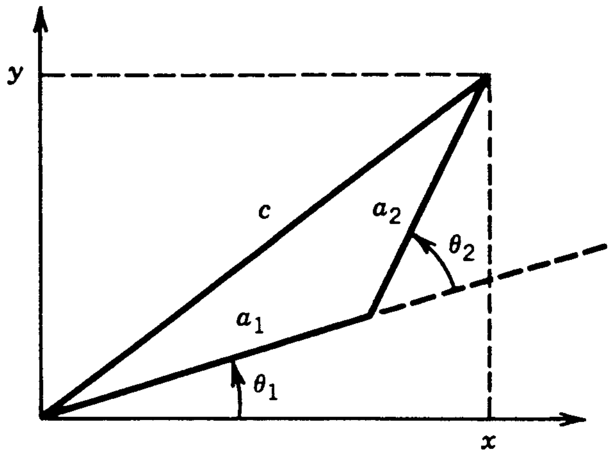

FIGURE 1-28
Solving for the joint angles of a two-link planar arm.

$$\dot{y} = a_1\cos\theta_1\cdot\dot{\theta}_1 + a_2\cos(\theta_1+\theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

Using the vector notation $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ we may write these equations as

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -a_1\sin\theta_1 - a_2\sin(\theta_1+\theta_2) & -a_2\sin(\theta_1+\theta_2) \\ a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) \end{bmatrix}\dot{\boldsymbol{\theta}} \\ &= J\dot{\boldsymbol{\theta}}\end{aligned} \tag{1.5.11}$$

The matrix J defined by (1.5.11) is called the **Jacobian** of the manipulator and is a fundamental object to determine for any manipulator. In Chapter Five we present a systematic procedure for deriving the Jacobian for any manipulator in the so-called **cross-product form.**

The determination of the joint velocities from the end-effector velocities is conceptually simple since the velocity relationship is linear. Thus the joint velocities are found from the end-effector velocities via the inverse Jacobian

$$\dot{\boldsymbol{\theta}} = J^{-1}\dot{\mathbf{x}} \tag{1.5.12}$$

or

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \frac{1}{a_1a_2\sin\theta_2}\begin{bmatrix} a_2\cos(\theta_1+\theta_2) & a_2\sin(\theta_1+\theta_2) \\ -a_1\cos\theta_1 - a_2\cos(\theta_1+\theta_2) & -a_1\sin\theta_1 - a_2\sin(\theta_1+\theta_2) \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \tag{1.5.13}$$

The determinant $\det J$ of the Jacobian in (1.5.11) is $a_1a_2\sin\theta_2$. The Jacobian does not have an inverse, therefore, when $\theta_2 = 0$ or π, in which case the manipulator is said to be in a **singular configuration**, such as shown in Figure 1-29 for $\theta_2 = 0$. The determination of such singular configurations is important for several reasons. At singular

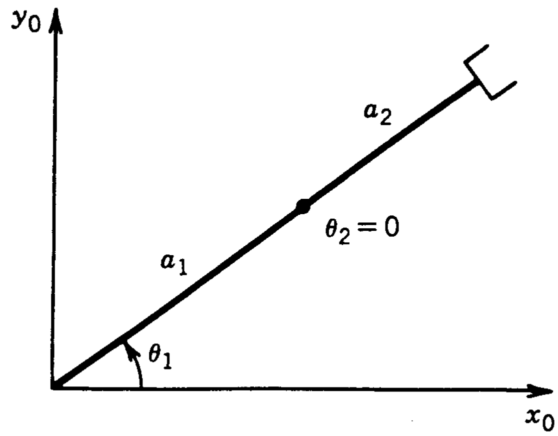

FIGURE 1-29
A singular configuration.

configurations there are infinitesimal motions that are unachievable; that is, the manipulator end-effector cannot move in certain directions. In the above cases the end effector cannot move in the direction parallel to a_1 from a singular configuration. Singular configurations are also related to the non-uniqueness of solutions of the inverse kinematics. For example, for a given end-effector position, there are in general two possible solutions to the inverse kinematics. Note that the singular configuration separates these two solutions in the sense that the manipulator cannot go from one configuration to the other without passing through the singularity. For many applications it is important to plan manipulator motions in such a way that singular configurations are avoided.

1.5.4 PROBLEM 4: DYNAMICS

A robot manipulator is basically a positioning device. To control the position we must know the dynamic properties of the manipulator in order to know how much force to exert on it to cause it to move. Too little force and the manipulator is slow to react. Too much force and the arm may crash into objects or oscillate about its desired position.

Deriving the dynamic equations of motion for robots is not a simple task due to the large number of degrees of freedom and nonlinearities present in the system. In Chapter Six we develop techniques based on Lagrangian dynamics for systematically deriving the equations of motion of such a system. In addition to the rigid links, the complete description of robot dynamics includes the dynamics of the actuators that produce the forces and torques to drive the robot, and the dynamics of the drive trains that transmit the power from the actuators to the links. Thus, in Chapter Seven we also discuss actuator and drive train dynamics and their effects on the control problem.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

a_i = distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

d_i = distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

α_i = the angle between z_{i-1} and z_i measured about x_i (See Figure 3-3).

θ_i = the angle between x_{i-1} and x_i measured about z_{i-1} (See Figure 3-3). θ_i is variable is joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.2.1).

Step 9: Form $T_0^n = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

3.3 EXAMPLES

In the D-H convention the only variable angle is θ, so we simplify notation by writing c_i for $\cos\theta_i$, etc. We also denote $\theta_1 + \theta_2$ by θ_{12}, and $\cos(\theta_1 + \theta_2)$ by c_{12} and so on. In the following examples it is important to remember that the D–H convention, while systematic, still allows considerable freedom in the choice of some of the manipulator parameters. This is particularly true in the case of parallel joint axes or when prismatic joints are involved.

(i) *Example 3.3.1 Planar Elbow Manipulator*

Consider the two-link planar arm of Figure 3-6. The joint axes z_0 and z_1 are normal to the page. We establish the base frame $o_0x_0y_0z_0$ as shown. The origin is chosen at the point of intersection of the z_0 axis with the page and the direction of the x_0 axis is completely arbitrary. Once the base frame is established, the $o_1x_1y_1z_1$ frame is fixed as shown by the D–H convention, where the origin o_1 has been located at the intersection of z_1 and the page. The final frame $o_2x_2y_2z_2$ is fixed by choosing the origin o_2 at the end of link 2 as shown. The link parameters are shown in Table 3-1. The A-matrices are determined from (3.2.1) as

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3.1}$$

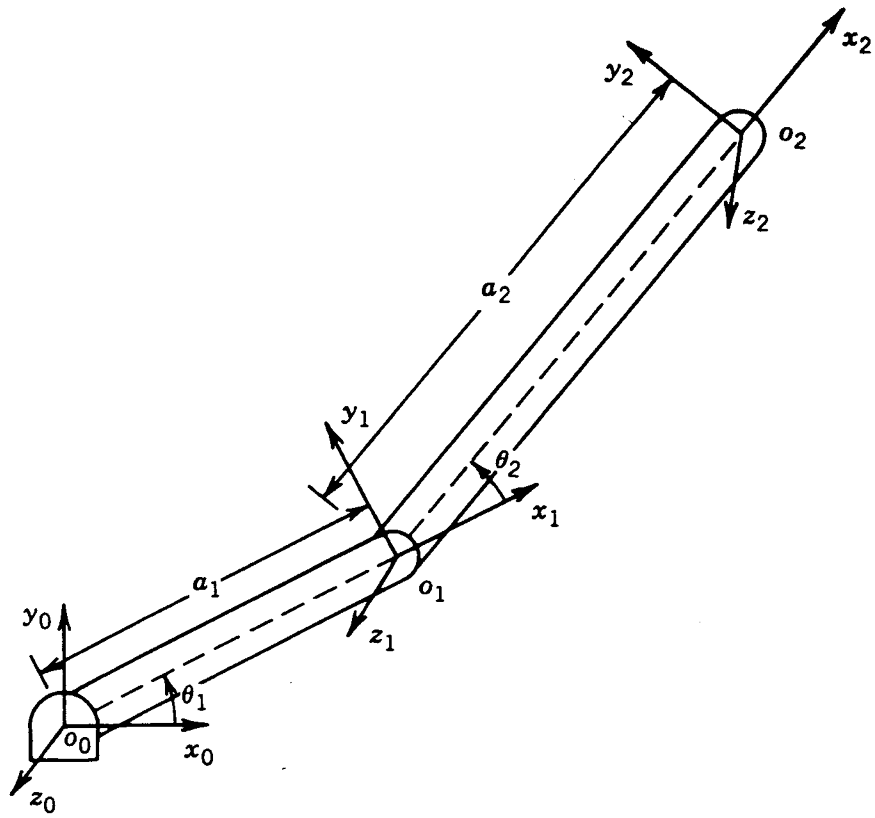

FIGURE 3-6
Two-link planar manipulator.

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3.2}$$

The T-matrices are thus given by

$$T_0^1 = A_1 \tag{3.3.3}$$

TABLE 3-1
Link Parameters for 2-link Planar Manipulator

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

variable