Sample Exam No. 2 Solutions

1

a) Kinetic Energy $K = 1/2M_1\dot{y}_1^2 + 1/2M_2\dot{y}_2^2$.

b) Potential Energy $P = 1/2K_1y_1^2 + 1/2K_2(y_2 - y_1)^2$.

c) $L = K - P = 1/2M_1\dot{y}_1^2 + 1/2M_2\dot{y}_2^2 - 1/2K_1y_1^2 - 1/2K_2(y_2 - y_1)^2$

$$\frac{\partial L}{\partial \dot{y}_1} = M_1\dot{y}_1; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} = M_1\ddot{y}_1; \quad \frac{\partial L}{\partial y_1} = -K_1y_1 + K_2(y_2 - y_1)$$

$$\frac{\partial L}{\partial \dot{y}_2} = M_2\dot{y}_2; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} = M_2\ddot{y}_2; \quad \frac{\partial L}{\partial y_2} = -K_2(y_2 - y_1).$$

Euler-Lagrange Equations

$$\begin{aligned} M_1\ddot{y}_1 + K_1y_1 - K_2(y_2 - y_1) &= -B_1\dot{y}_1 \\ M_2\ddot{y}_2 + K_2(y_2 - y_1) &= u - B_2\dot{y}_2 \end{aligned}$$

or

$$\begin{aligned} M_1\ddot{y}_1 + B_1\dot{y}_1 + K_1y_1 - K_2(y_2 - y_1) &= 0 \\ M_2\ddot{y}_2 + B_2\dot{y}_2 \qquad + K_2(y_2 - y_1) &= u. \end{aligned}$$

2

$$\begin{aligned} \ddot{\theta}_1 + (3sin\theta_1)\ddot{\theta}_2 + \dot{\theta}_1\dot{\theta}_2 + sin\theta_2 &= \tau_1 + \tau_2 \\ \ddot{\theta}_2 - (2sin\theta_1)\ddot{\theta}_1 - \dot{\theta}_1\dot{\theta}_2 &= \tau_2. \end{aligned}$$

In matrix form we can write this as

$$M(\theta)\ddot{\theta} + h(\theta, \dot{\theta}) = B\tau$$

where

$$M(\theta) = \begin{bmatrix} 1 & 3sin\theta_1 \\ -2sin\theta_1 & 1 \end{bmatrix}$$

$$h(\theta_1\dot{\theta}) = \begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 + sin\theta_2 \\ -\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Therefore, by inspection, we see that an inverse dynamics control law is

$\tau = B^{-1}\{M(\theta)a + h(\theta,\dot{\theta})\}$.

Carrying out the calculations leads to

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\left\{\begin{bmatrix}1 & 3sin\theta_1\\ -2sin\theta_1 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} + \begin{bmatrix}\dot{\theta}_1\dot{\theta}_2 + sin\theta_2\\ -\dot{\theta}_1\dot{\theta}_2\end{bmatrix}\right\}$$

or

$$\tau_1 = (1 + 2sin\theta_1)a_1 + (3sin\theta_1 - 1)a_2 + 2\dot{\theta}_1\dot{\theta}_2 + sin\theta_1$$

$$\tau_2 = -(2sin\theta_1)a_1 + a_2 - \dot{\theta}_1\dot{\theta}_2.$$

b) For

$$\ddot{\theta}_1 = a_1$$

$$\ddot{\theta}_2 = a_2$$

we can use

$$a_1 = -2\varsigma_1\omega_1\dot{\theta}_1 - \omega_1^2\theta_1 + r_1$$

$$a_2 = -2\varsigma_2\omega_2\dot{\theta}_2 - \omega_2^2\theta_2 + r_2$$

where r_1, r_2 are reference input signals. With $\varsigma_1 = \varsigma_2 = 1$; $\omega_1 = \omega_2 = 10$ we have

$$a_1 = -20\dot{\theta}_1 - 100\theta_1 + r_1$$

$$a_2 = -20\dot{\theta}_2 - 100\theta_2 + r_2.$$

3 Without the feedforward transfer function $F(s)$ the closed loop system is $\frac{GH}{1+GH}$ where $G(s) = \frac{1}{s^2+4s}$ and $H(s)$ is a PD compensator $H(s) = K_p + K_v s$. Carrying out the calculations leads to

$$\frac{GH}{1+GH} = \frac{K_p + K_v s}{s^2 + (4 + K_v)s + K_p}.$$

a) The closed loop characteristic polynomial is $s^2 + (4 + K_v)s + K_p$. Therefore

$$K_p = \omega^2$$

$$4 + K_v = 2\varsigma\omega$$

with $\varsigma = 1$, $\omega = 12$ this leads to

$$K_p = 144$$

$$K_v = 20.$$

b) To track a time-varying reference we know that we should choose

$$F(s) = \frac{1}{G(s)} = s^2 + 4s.$$

This leads to the feedforward signal

$$\left(\frac{d^2}{dt^2} + 4\frac{d}{dt}\right) cos3t$$

$$= -9cos3t - 12sin3t.$$

4

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_1^2 x_2.$$

a) The equilibrium points are found from

$$\begin{aligned} x_2 &= 0 \\ -x_1 - x_1^2 x_2 &= 0 \end{aligned} \quad \Rightarrow \quad \binom{x_1}{x_2} = \binom{0}{0}.$$

b) With $V = x_1^2 + x_2^2$ we have

$$\begin{aligned} \dot{V} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1x_2 + 2x_2(-x_1 - x_1^2 x_2)\,. \\ &= -2x_1^2x_2^2 < 0 \end{aligned}$$

Therefore $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \binom{0}{0}$ and $\dot{V}(\mathbf{x}) < 0$.

Hence the origin is an asymptotically stable equilibrium.