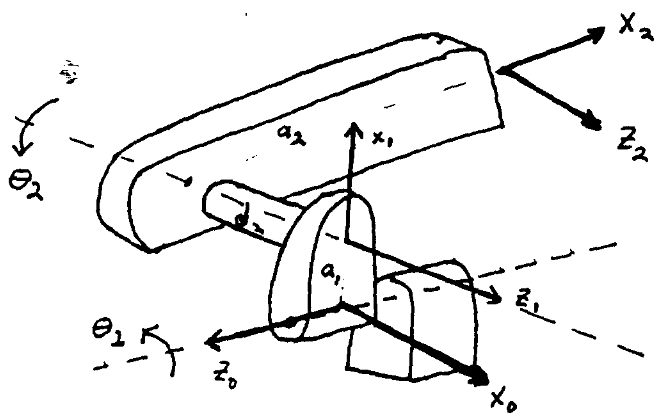
Sample Exam No. 1 Solutions

1 a) The workspace is a two–dimensional curved surface.

b)



link	a	α	d	θ
1	a_1	90	0	θ_1
2	a_2	0	d_2	θ_2

c)

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & a_1c_1 \\ s_1 & 0 & -c_1 & a_1s_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^2 = A_1A_2 = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & a_1c_1 + s_1d_2 + a_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & a_1s_1 - c_1d_2 + a_2s_1c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

d) Since both joints are revolute

$$J = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; \quad z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix} ; \quad z_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad o_1 = \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix}; \quad o_2 = \begin{bmatrix} a_1c_1 + s_1d_2 + +a_2c_1c_2 \\ a_1s_1 - c_1d_2 + a_2s_1c_2 \\ a_2s_2 \end{bmatrix}$$

$$z_0 \times (o_2 - o_0) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ a_1c_1 + s_1d_2 + a_2c_1c_2 & a_1s_1 - c_1d_2 + a_2s_1c_2 & a_2s_2 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1s_1 + c_1d_2 - a_2s_1c_2 \\ a_1c_1 + s_1d_2 + a_2c_1c_2 \\ 0 \end{bmatrix}$$

$$z_1 \times (o_2 - o_1) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ s_1 & -c_1 & 0 \\ s_1d_2 + a_2c_1c_2 & -c_1d_2 + a_2s_1c_2 & a_2s_2 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2c_1s_2 \\ -a_2s_1s_2 \\ a_2c_2 \end{bmatrix}.$$

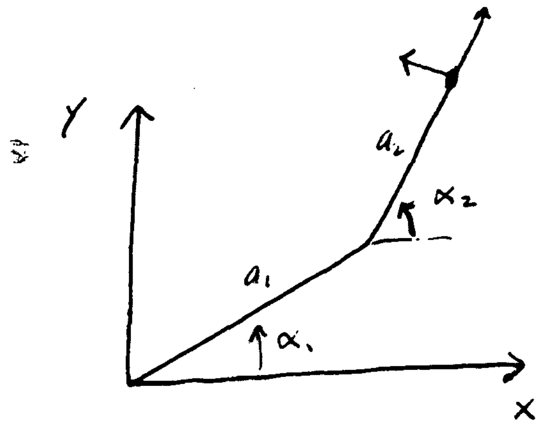
Therefore

$$J = \begin{bmatrix} -a_1s_1 + c_1d_2 - a_2s_1c_2 & -a_2c_1s_2 \\ a_1c_1 + s_1d_2 + a_2c_1c_2 & -a_2s_1s_2 \\ 0 & a_2c_2 \\ 0 & s_1 \\ 0 & -c_1 \\ 1 & 0 \end{bmatrix}$$

J always has rank 2. The easiest way to see this is to look at the last three rows

$$\begin{bmatrix} 0 & s_1 \\ 0 & -c_1 \\ 1 & 0 \end{bmatrix} = [\, z_0 \quad z_1 \,].$$

These two vectors are always independent since s_1, c_1 can never be zero simultaneously. Physically this means that z_0, z_1 can never be parallel. Therefore there are no singularities for this manipulator.



The direct approach is to write

$$x = a_1c_1 + a_2c_2 \quad \text{where} \quad s_i sin\alpha_i$$
$$y = a_1s_1 + a_2s_2 \qquad\qquad c_i = cos\alpha_i$$

and differentiate to obtain

$$\dot{x} = -a_1s_1\dot{\alpha}_1 - a_2s_2\dot{\alpha}_2$$
$$\dot{y} = a_1c_1\dot{\alpha}_1 + a_2c_2\dot{\alpha}_2.$$

Also, both z-axis are parallel so the angular velocity is just $\dot{\theta}_1 + \dot{\theta}_2 = \dot{\alpha}_2$.

The Jacobian is therefore

$$\mathbf{J}(\alpha) = \begin{bmatrix} -a_1s_1 & -a_2s_2 \\ a_1c_1 & a_2c_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Second solution: From the text we have

$$J(\theta) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

where $s_i = cos\theta_i$, etc.

The transformation between θ_i and α_i is

$$\begin{aligned} \alpha_1 &= \theta_1 \\ \alpha_2 &= \theta_1 + \theta_2 \end{aligned} \quad \text{or} \quad \alpha = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \theta.$$

The inverse transformation is

$$\theta = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \alpha.$$

Therefore

$$\dot{x} = \mathbf{J}(\alpha)\dot{\alpha} = J(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \alpha) \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \dot{\alpha}$$

Computing

$$J(\theta) \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & +a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 & -a_2 s_{12} \\ a_1 c_1 & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, since

$$sin\theta_1 = sin\alpha_1 \qquad sin(\theta_1 + \theta_2) = sin\alpha_2$$
$$cos\theta_1 = cos\alpha_1 \qquad cos(\theta_1 + \theta_2 = cos\alpha_2$$

$$\mathbf{J}(\alpha) = \begin{bmatrix} -a_1 sin\alpha_1 & -a_2 sin\alpha_2 \\ a_1 cos\alpha_1 & a_2 cos\alpha_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3

a) nonsingular: rank $J = 6$.

b) singular: rank $J = 3$

- elbow straight
- wrist axes aligned
- wrist center intersects z_0.

c) singular: rank $J = 4$
- elbow straight
- wrist center intersects z_0.

d) singular rank $J = 4$
- elbow straight
- wrist axes aligned.

4 Use the equivalent axis/angle representation

$R_1 = R_{k_1,\theta}$; $r_2 = R_{k_2,\theta_2}$.

If $k_1 = k_2 = k$, i.e., if the axis of rotation are the sa e, then

$R_{k_1,\theta_1} R_{k_2,\theta_2} = R_{k_1,(\theta_1+\theta_2)} = R_{k,}$) $= R_{k,\theta_2} R_{k,\theta_1} = R_{k_2\theta_2} R_{k_1,\theta_1}$

Thus $R_1 R_2 = R_2 R_1$.

Conjecture: This is the only way that two rotations can com er words we conjecture the following

Theorem: Let $R_1, R_2 \in SO(3)$. Then $R_1 R_2 = R_2 R_1$ if and only if R_1 and R_2 are rotations about the same equivalent axis.

This conjecture turns out to be true. We have already proved half of it mely if R_1, R_2 are rotations about the same axis then
that is, if $R_1 R_2 = R_2 R_1$ then R_1 and e same

Proof: We are given 2 facts:

1) $R_1, R_2 \in SO(3)$ and 2) $R_1 R_2 = R_2 R_1$.

Let $R_1 = R_{k_1,\theta_1}$; $R_2 = R_{k_2,\theta_2}$.

The first step is to show that

$R_1 R_2 = R_2 R_1 \Rightarrow S(k_1) S(k_2) = S(k_2) S(k_1)$

where $S(k_1), S(k_2)$ are skew symmetric. This can be shown using

$R_{k,\theta} = I + S(k) sin\theta + S^2(k)\ vers\ \theta$.

The second step is to show that

$S(k_1) S(k_2) = S(k_2) S(k_1) \Rightarrow k_1 = k_2$.

This can be shown by multiplying out the matrices and equating coefficients. The result is

$$R_1R_2 = R_2R_1 \Rightarrow k_1 = k_2$$

which is the "only if" part of the theorem.