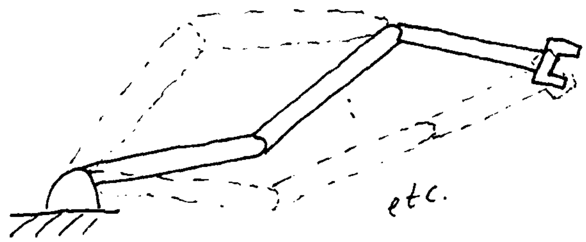
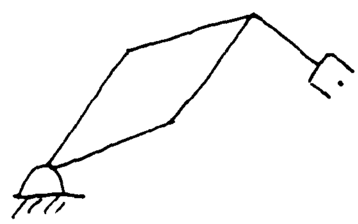
CHAPTER 4

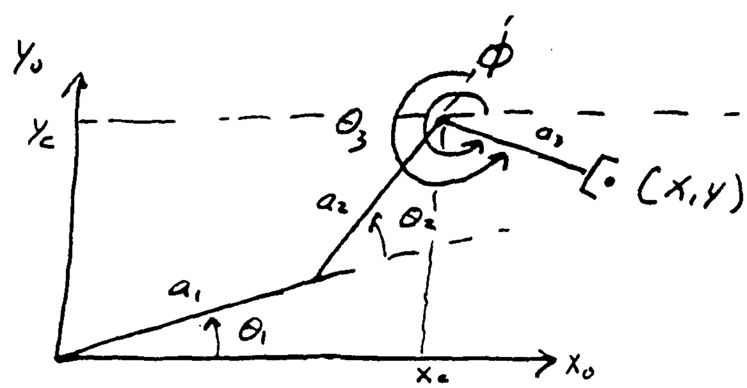
4–1



Given a desired position of the end-effector there are infinitely many solutions of the inverse kinematics problem



If the orientation of the end-effector is fixed then there are 2 solutions in general.



Suppose we are given position (x, y) and orientation ϕ of the end-effector

Then given the "wrist center" (x_c, y_c) the solutions for (θ_1, θ_2) is given by the 2-link planar arm solution

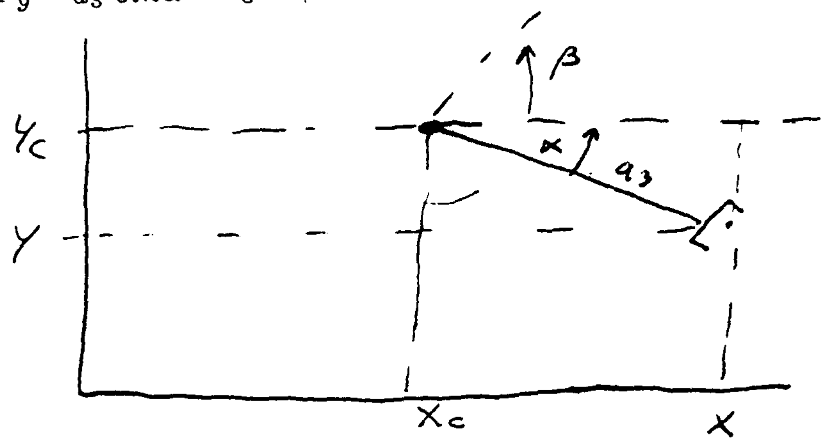
$$\theta_2 = Atan\ (D_1 \pm \sqrt{1-D^2}); \theta_1 = Atan\ (x_c, y_c) - Atan\ (a_1 + a_2 c_2, a_2 s_2)$$

where

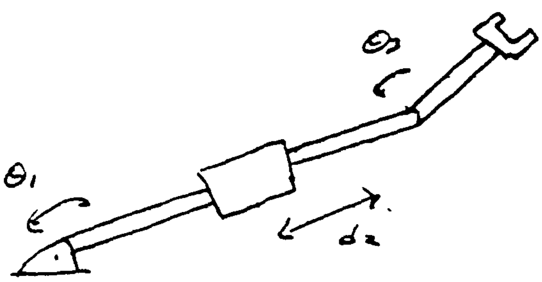
$$D = \frac{x_c^2 + y_c^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

To find (x_c, y_c) and θ_3 note from the diagram below that

$$x_c = x - a_3\ cos\alpha \qquad \alpha = -\phi$$
$$y_c = y - a_3\ sin\alpha \qquad \theta_3 = \phi - \theta_1 - \theta_2$$

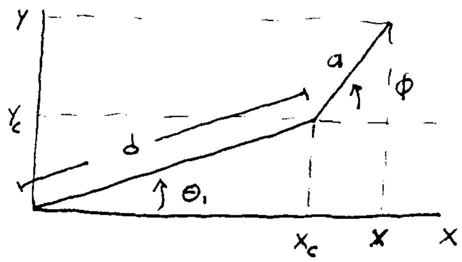


4–2

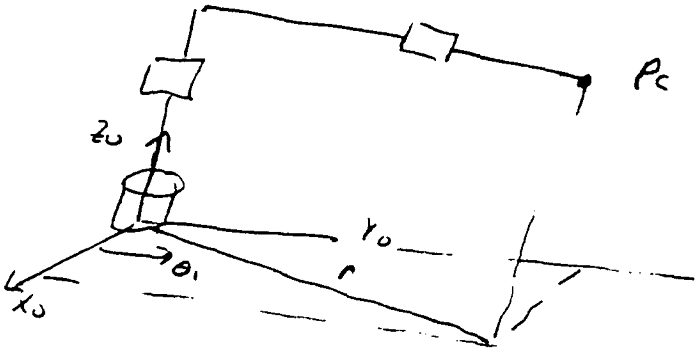


Again, there are infinitely many solutions for a given end-effector position (x, y). If the orientation ϕ is fixed then

$x_c = x - a\ cos\phi$
$y_c = y - a\ sin\phi$
$d = \sqrt{x_c^2 + y_c^2}$
$\theta_1 = tan^{-1}(y_c/x_c)$



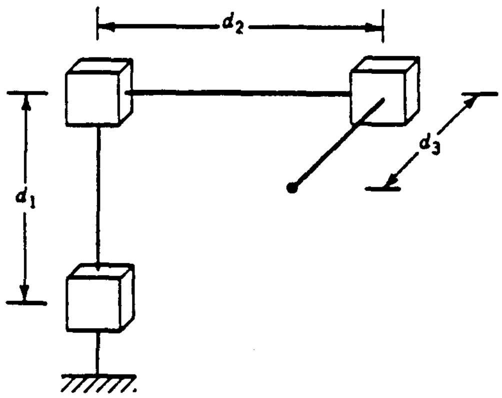
4–3



Given $\mathbf{p}_c = (x_c, y_c, z_c)^T$, we have from the diagram above

$\theta_1 = Atan\ (x_c, y_c)$
$d_2 = z_c - 1m$
$d_3 = r - 1m \quad r = \sqrt{x_c^2 + y_c^2}$

4–4



given $d = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$ can be reached by setting

$$\begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}$$

4–5

$$R_3^6 = (R_0^3)^T R = U = \begin{bmatrix} r_{31} & r_{32} & r_{33} \\ r_{11}c_1 + r_{21}s_1 & r_{12}c_1 + r_{22}s_1 & r_{13}c_1 + r_{23}s_1 \\ -r_{11}s_1 + r_{21}c_1 & -r_{11}s_1 + r_{22}c_1 & -r_{13}s_1 + r_{23}c_1 \end{bmatrix}$$

I. If not both $u_{13} + u_{23}$ are zero, then

$\theta_5 = Atan(-r_{13}s_1 + r_{23}c_1, \pm\sqrt{1 - (-r_{13}s_1 + r_{23}c_1)^2})$

a) If the positive square root is chosen

$\theta_4 = Atan(r_{33}, r_{13}c_1 + r_{23}s_1)$

$\theta_6 = Atan(+r_{11}s_1 - r_{21}c_1, -s_1 r_{12} + c_1 r_{22})$

b) If the negative square root is chosen

$\theta_4 = Atan(-r_{33}, -r_{13}c_1 - r_{23}s_1)$

$\theta_6 = Atan(-r_{11}s_1 + r_{21}c_1, s_1 r_{12} - c_1 r_{22})$

II. If $u_{13} = u_{23} = 0$

a) If $u_{33} = 1$

$0 = r_{33} = r_{13}c_1 + r_{28}s_1 = c_4 s_5 = s_4 s_5 \Rightarrow s_5 = 0 \quad \theta_5 = 0°$

$\theta_4 + \theta_6 = Atan(r_{31}, r_{11}c_1 + r_{21}s_1) = Atan(r_{31}, -r_{32})$

b) If $u_{33} = -1 \quad \theta_4 = 0; \quad c_5 = -1 \quad s_5 = 0 \quad \theta_5 = \pi$

$\theta_4 - \theta_6 = Atan(-r_{31}, -r_{32}) = Atan(-r_{11}c_1 - r_{21}s_1, -r_{12}c_1 - r_{22}s_1)$

4–6

link	a_i	α_i	d_i	θ_i
1	0	-90	d_1^*	0
2	0	90	d_2^*	90
3	0	0	d_3^*	0
4	0	90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* denotes variable

$$R_0^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Given d and R $\quad p_c = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}$

$$R_3^6 = (R_0^3)^T\ R = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -r_{31} & -r_{32} & -r_{33} \\ r_{21} & r_{22} & r_{23} \\ r_{11} & r_{12} & r_{13} \end{bmatrix}$$

Equate R_3^6 to matrix (4.4.1).

Suppose that r_{33} and r_{23} are nonzero, then $r_{13} \neq \pm 1$, so

$$c_\theta = r_{13}, \quad s_\theta = \pm\sqrt{1 - r_{13}^2} \text{ and } \theta = Atan\ (r_{13},\ \sqrt{1 - r_{13}^2})$$

if $s_\theta > 0$, choose $\phi = Atan\ (-r_{33}, r_{23})$ and $\psi = Atan\ (-4_{11}, r_{12})$

However, if $r_{33} = r_{23} = 0$, then $r_{13} = \pm 1$

if $r_{13} = +1 \quad \theta = 0, \quad \phi + \psi = Atan(-r_{31}, r_{21}) = Atan(-r_{31}, r_{32})$

if $r_{13} = -1 \quad \theta = \pi, \quad \phi - \psi = Atan(r_{31}, r_{32}) = Atan(-r_{21}, -r_{22})$

if $r_{13} = \pm 1$, there are an infinite number of solutions.

4–8

a) $\mathbf{p}_c = \mathbf{d} - d_6 R\mathbf{k} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} - d_6 \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$

where

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

b) Solution I

$$\theta_1 = Atan(p_{cx}, p_{cy})$$

$$r = \sqrt{p_{cx}^2 + p_{cy}^2}$$

$$l^2 = r^2 + p_{cz}^2 = d_3^2 + d_2^2$$

$$d_3 = \sqrt{r^2 + p_{cz}^2 - d_2^2}$$

$$\theta_2 = \phi - \alpha = Atan(r, p_{cz}) - Atan(d_2, d_3)$$

Solution II

$$\theta_1 = \pi + Atan(p_{cx}, p_{cy})$$

$$r = \sqrt{p_{cx}^2 + p_{cy}^2}$$

$$l^2 = r^2 + p_{cz}^2 = d_3^2 + d_2^2$$

$d_3 = \sqrt{r^2 + p_{cz}^2 - d_2^2}$

$\theta_2 = \pi - \phi - \alpha$

$\theta_2 = \pi - Atan(r, p_{cz}) - Atan(d_2, d_3)$

c) $R_0^3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 \\ s_1c_2 & c_1 & s_1s_2 \\ -s_2 & 0 & c_2 \end{bmatrix}$ found by multiplying A_1 A_2 A_3 and extracting first 3 rows and columns.

$R_3^6 = (R_0^3)^T R =$

$$\begin{bmatrix} c_1c_2r_{11} + s_1c_2r_{21} - s_2r_{31} & c_1c_2r_{12} + s_1c_2r_{22} - s_2r_{32} & c_1c_2r_{13} + s_1c_2r_{23} - s_2r_{33} \\ -s_1r_{11} + c_1r_{21} & -s_1r_{12} + c_1r_{22} & -s_1r_{13} + c_1r_{23} \\ c_1s_2r_{11} + s_1s_2r_{21} + c_2r_{31} & c_1s_2r_{12} + s_1s_2r_{22} + c_2r_{32} & c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33} \end{bmatrix}$$

Assume $r_{13} \neq 0$ and $R_{23} \neq 0$ then

$c_5 = c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33}$ and

$s_5 = \pm\sqrt{1 - (c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33})^2}$

if $s_5 > 0$ then

$\theta_5 = Atan(c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33}, \sqrt{1 - (c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33})^2})$

$\theta_4 = Atan(c_1c_2r_{13} + s_1c_2r_{23} - s_2r_{33}, -s_1r_{13} + c_1r_{23})$

$\theta_6 = Atan(c_1s_2r_{11} + s_1s_2r_{21} + c_2r_{31}, -c_1s_2r_{12} - s_1s_2r_{22} - c_2r_{32})$

if $s_5 < 0$ then

$\theta_5 = Atan(c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33}, -\sqrt{1 - (c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33})^2})$

$\theta_4 = Atan(-c_1c_2r_{13} - s_1c_2r_{23} + s_2r_{33}, s_1r_{13} - c_1r_{23})$

$\theta_6 = Atan(-c_1s_2r_{11} - s_1s_2r_{21} - c_2r_{31}, c_1s_2r_{12} + s_1s_2r_{22} + c_2r_{32})$

if $r_{13} = r_{23} = 0$ then $r_{33} = \pm 1$

if $r_{33} = +1 \quad \theta_5 = \theta_2$ and $\theta_4 = \pi$

if $s_5 > 0 \quad \theta_6 = Atan(c_1s_2r_{11} + s_1s_2r_{21} + c_2r_{31}, -c_1s_2r_{12} - s_1s_2r_{22} - c_2r_{32})$

if $s_5 < 0 \quad \theta_6 = Atan(-c_1s_2r_{11} - s_1s_2r_{21} - c_2r_{31}, c_1s_2r_{12} + s_1s_2r_{22} + c_2r_{32})$

if $r_{33} = -1 \quad \theta_5 = \pi - \theta_2$ and $\theta_4 = 0$

if $s_5 > 0$ $\quad \theta_6 = Atan(c_1 s_2 r_{11} + s_1 s_2 r_{21} + c_2 r_{31}, -c_1 s_2 r_{12} - s_1 s_2 r_{22} - c_2 r_{32})$

if $s_5 < 0$ $\quad \theta_6 = Atan(-c_1 s_2 r_{11} - s_1 s_2 r_{21} - c_2 r_{31}, c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32})$

There are 4 solutions available where 2 are from the inverse position kinematics and 2 come from the wrist inverse kinematics.

4–9

link	a_i	α_i	d_i	θ_i
1	0	90°	d_1	θ_1^*
2	a_2	0	d_2	θ_2^*
3	a_3	0	0	θ_3^*

a)

$x_c = x_0 - d_6 c_5 c_1$

$y_c = y_0 - d_6 c_5 s_1$

$z_c = z_0 - z_{0c}$

b)

$\theta_1 = \phi - \alpha$

$\phi = tan(\frac{y_c}{x_c})$

$\alpha = tan(\frac{a_3 c_{23} + a_2 c_2}{d_2})$

Elbow Right

$$\theta_1 = tan(\frac{y_c}{x_c}) - tan(\frac{a_3 c_{23} + a_2 c_2}{d_2})$$

$\theta_1 = \phi + \alpha$

$\phi = tan(\frac{y_c}{x_c})$

$\alpha = tan(\frac{a_3 c_{23} + a_2 c_2}{d_2})$

Elbow Left

$$\theta_1 = tan(\frac{y_c}{x_c}) + tan(\frac{a_3 c_{23} + a_2 c_2}{d_2})$$

by the 2-link planar solution

$$\theta_3 = Atan(d, \pm\sqrt{1 - D^2}) \quad \text{where} \quad D = \frac{s_c^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2 a_3}$$

$$\theta_2 = Atan(s_c, z_c - d_1) - Atan(a_2 + a_3 c_{31} a_3 s_3)$$

c) $R_0^3 = \begin{bmatrix} c_1c_2c_3 - c_1s_2s_3 & -c_1c_2s_3 - c_1s_2c_3 & s_1 \\ s_1c_2c_3 - s_1s_2s_3 & -s_1c_2s_3 - s_1s_2c_3 & -c_1 \\ s_2c_3 + c_2s_3 & -s_2s_3 + c_2c_3 & 0 \end{bmatrix}$

$$u = (R_0^3)^T R = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$$

$$u_{11} = r_{11}(c_1c_2c_3 - c_1s_2s_3) + r_{21}(s_1c_2c_3 - s_1s_2s_3) + r_{31}(s_2c_3 + c_2s_3)$$
$$u_{21} = -r_{12}(c_1c_2s_3 + c_1s_2c_3) - r_{21}(s_1c_2s_3 + s_1s_2c_3) + r_{31}(-s_2s_3 + c_2c_3)$$
$$u_{31} = r_{11}s_1 - r_{21}c_1$$
$$u_{12} = r_{12}(c_1c_2c_3 - c_1s_2s_3) + r_{22}(s_1c_2c_3 - s_1s_2s_3) + r_{32}(s_2c_3 + c_2s_3)$$
$$u_{22} = -r_{12}(c_1c_2s_3 + c_1s_2c_3) - r_{22}(s_1c_2s_3 + s_1s_2c_3) + r_{32}(-s_2s_3 + c_2c_3)$$
$$u_{32} = r_{12}s_1 - r_{22}c_1$$
$$u_{13} = r_{13}(c_1c_2c_3 - c_1s_2s_3) + r_{23}(s_1c_2c_3 - s_1s_2s_3) + r_{33}(s_2c_3 + c_2s_3)$$
$$u_{23} = -r_{13}(c_1c_2s_3 + c_1s_2c_3) - r_{23}(s_1c_2s_3 + s_1s_2c_3) + r_{33}(-s_2s_3 + c_2c_3)$$
$$u_{33} = r_{13}s_1 - r_{23}c_1$$

Inverse orientation solutions

I. Suppose not both u_{13}, u_{23} are zero

$\theta_5 = Atan(u_{33} \pm \sqrt{1 - u_{33}^2})$

a) If the positive square root is chosen

$\theta_4 = Atan(u_{13}, u_{23})$
$\theta_6 = Atan(u_{31}, u_{32})$

b) If the negative square root is chosen

$\theta_4 = Atan(-u_{13}, -u_{23})$
$\theta_6 = Atan(u_{31}, -u_{32})$

II. If $u_{13} = u_{23} = 0$

a) And if $u_{33} = 1$; $\quad \theta_5 = 0$

$\theta_4 + \theta_6 = Atan(u_{11}, u_{21})$

b) Or if $u_{33} = -1$; $\quad \theta_5 = \pi$

$\theta_4 - \theta_6 = Atan(-u_{11}, -u_{12})$

4–10 Using Figure 3–21 and given $\mathbf{d}$ and

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$\mathbf{p}_c = \mathbf{d} - d_5 \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$ but $d_5 = 5.125$ so $\mathbf{p}_c = \mathbf{d} - 5.125 \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$. There are two solutions for θ_1 : $\theta_1 = Atan(p_{cx}, p_{cy})$ and $\theta_1 = \pi + Atan(p_{cx}, p_{cy})$.

Once θ_1 is chosen, θ_2 and θ_3 reduce to the 2 link manipulator solved in Chapter One.

I. $\theta_1 = Atan(p_{cx}, p_{cy})$

define $r = \sqrt{p_{cx}^2 + p_{cy}^2}$ and $l^2 = p_{cx}^2 + p_{cy}^2 + p_{cz}^2$ and $D = c_3 = l^2 - a_2^2 - a_3^2/2a_2a_3$ but $a_2 = a_3 = 9$

$D = l^2 - 162/162 = l^2/162 - 1 \quad s_3 = \pm\sqrt{1 - (l^2/162 - 1)^2} = \pm l/9\sqrt{1 - l^2/324}$

For elbow down $\theta_3 = Atan(l^2/162 - 1, l/9\sqrt{1 - l^2/324})$

$$\begin{aligned} \theta_2 &= Atan(r, p_{cz}) - Atan(a_2 + a_3c_3, a_3s_3) \\ &= Atan(r, p_{cz}) - Atan(9(1 + c_3), 9s_3) \\ &= Atan(r, p_{cz}) - Atan(l^2/18, l\sqrt{1 - l^2/324}) \end{aligned}$$

For elbow up, $\theta_3 = Atan(l^2/162 - 1, -l/9\sqrt{1 - l^2/324})$

$$\begin{aligned} \theta_2 &= Atan(r, p_{cz}) + Atan(9(1 + c_3), 9s_3) \\ &= Atan(r, p_{cz}) + Atan(l^2/18, -l\sqrt{1 - l^2/324}) \end{aligned}$$

II. $\theta_2 = \pi + Atan(p_{cx}, p_{cy})$

For elbow down, $\theta_3 = Atan(l^2/162 - 1, l/9\sqrt{1 - l^2/324})$ but

$$\begin{aligned} \theta_2 &= \pi - Atan(r, p_{cz}) + Atan(9(1 + c_3), 9s_3) \\ &= \pi - Atan(r, p_{cz}) + Atan(l^2/18, l\sqrt{1 - l^2/324}) \end{aligned}$$

For elbow up, $\theta_3 = Atan(l^2/162 - 1, -l/9\sqrt{1 - l^2/324})$

$$\begin{aligned} \theta_2 &= \pi - Atan(r, p_{cz}) - Atan(9(1 + c_3), 9s_3) \\ &= \pi - Atan(r, p_{cz}) - Atan(l^2/18, -l\sqrt{1 - l^2/324}) \end{aligned}$$

4–11

$u = R_0^{3T} R =$

$$\begin{bmatrix} c_1c_{23}r_{11} + s_1c_{23}r_{21} - s_{23}r_{31} & c_1c_{23}r_{12} + s_1c_{23}r_{22} - s_{23}r_{32} & c_1c_{23}r_{13} + s_1c_{23}r_{23} - s_{23}r_{33} \\ -c_1s_{23}r_{11} - s_1s_{23}r_{21} - c_{23}r_{31} & -c_1s_{23}r_{12} - s_1s_{23}r_{22} - c_{23}r_{32} & -c_1s_{23}r_{13} - s_1s_{23}r_{23} - c_{23}r_{33} \\ -s_1r_{11} + c_1r_{21} & -s_1r_{12} + c_1r_{22} & -s_1r_{13} + c_1r_{23} \end{bmatrix}$$

If $u_{13} = u_{23} = 0$ and $u_{33} = 1$

$\theta_4 + \theta_6 = Atan(c_1c_{23}r_{11} + s_1c_{23}r_{21} - s_{23}r_{31}, -c_1s_{23}r_{11} - s_1s_{23}r_{21} - c_{23}r_{31})$

If $u_{33} = -1$

$\theta_4 - \theta_6 = Atan(-c_1c_{23}r_{11} - s_1c_{23}r_{21} + s_{23}r_{31}, -c_1c_{23}r_{12} - s_1c_{23}r_{22} + s_{23}r_{32})$

4–12 For left arm solution: Equation 4.3.11 becomes

$\theta_1 = \phi - \alpha = Atan(p_x, p_y) - Atan(\sqrt{r^2 - a_1^2}, a_1)$

where

$r = \sqrt{p_x^2 + p_y^2}$

and Equation 4.3.12 becomes

$\theta_2 = Atan(\sqrt{r^2 - a_1^2}, s)$

where $s = p_z - d_1$

For right arm solution:

$\theta_1 = 2\pi - \gamma + \phi = \phi - \gamma$

where $\gamma = Atan(a_1, \sqrt{r^2 - a_1^2})$ so that Equation 4.3.11 becomes

$\theta_1 = Atan(p_x, p_y) - Atan(a_1, \sqrt{r^2 - a_1^2})$ and Equation 4.3.12 becomes

$\theta_2 = \pi + Atan(\sqrt{r^2 - a_1^2}, s)$

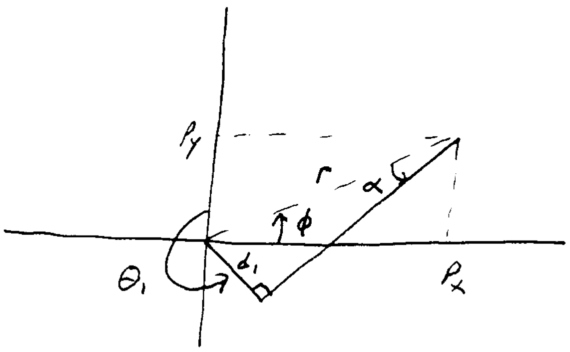
Note: Equation 4.3.6 is incorrect in the argument of the *Atan* function and should be corrected as follows:

(4.3.6) $\gamma = Atan(\sqrt{r^2 - d_1^2}, d_1) = Atan(\sqrt{p_x^2 + p_y^2 - d_1^2}, d_1)$

Equation 4.37 also needs to be modified in the following way.

(4.3.7) $\theta_1 = Atan(p_x, p_y) - Atan(d_1, \sqrt{p_x^2 + p_y^2 - d_1^2})$

For consistency, θ_1 for the right arm solution should be measured from the same zero configuration as the left arm, so that Figure 4–7 becomes



Using this definition for θ_1, Equation 4.3.7 becomes

$\theta_1 = Atan(p_x, p_y) - Atan(-\sqrt{r^2 - d_1^2}, d_1)$.