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Robotics Homework #2

Kostyantyn Vovk [0416794]

Question 2.3

excellent

$R_3 = R_1 * R_2$

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$

The determinant of the above matrix, expanded by row 3 is :

$$\begin{aligned} \text{Det} &= R_{13}\,(R_{12}\,R_{23} - R_{13}\,R_{22}) - (R_{11}\,R_{22} - R_{13}\,R_{21})\,R_{32} + (R_{11}\,R_{22} - R_{12}\,R_{21})\,R_{33} \\ &= R_{31}{}^2 + R_{32}{}^2 + R_{33}{}^2 = (R_{31}\ \ R_{32}\ \ R_{33})^2 = 1 \end{aligned}$$

Question 2.4

The transformation matrix for $R_{z,0}$ is the following

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can clearly see that the above matrix is the identity matrix, thus :

$\therefore R_{z,0} = I$

The transformation matrix for $R_{z,\theta}$ is the following :

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And the transformation matrix for $R_{z,\phi}$ is the following

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying the two above matrices gives us :

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & 0 \\ \sin(\theta+\phi) & \cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can notice that the above result is equivalent to $R_{z,\theta+\phi}$

$(\therefore R)_{z,\theta}\, R_{z,\phi} = R_{z,\theta+\phi}$

From the above identity which we proved it follows that :

$\mathbf{R}_{z,\theta}\ \mathbf{R}_{z,-\theta} = \mathbf{R}_{z,(\theta-\theta)} = \mathbf{R}_{z,0} = \mathbf{I}$

$\therefore\ \mathbf{R}_{(z,\theta)^{-1}} = \mathbf{R}_{z,-\theta}$

Question 2.5

Since the Rotation can be expressed as the following dot products

$\mathbf{i}_0\,\mathbf{i}_1 = \mathbf{1};$
$\mathbf{j}_0\,\mathbf{j}_1 = \mathbf{cos}\,\theta;$
$\mathbf{k}_0\,\mathbf{k}_1 = \mathbf{cos}\,\theta;$
$\mathbf{k}_0\,\mathbf{j}_1 = \mathbf{sin}\,\theta;$
$\mathbf{j}_1\,\mathbf{k}_1 = -\,\mathbf{sin}\,\theta;$

So the transformation matrix for $\mathbf{R}_{x,\theta}$ is :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Following the same procedure for $\mathbf{R}_{y,\theta}$ we get a transformation matrix of :

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Question 2.6

We have the following matrix A :

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

From Cramer' s rule we get the inverse of the matrix

$$\mathbf{A}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The determinant of the matrix A can be then simply found as :

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

$\therefore$ we have shown that A is indeed :

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Check proof in page 7 of Key

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Start with arbitrary values to prove the relation

Question 2.7

The total result can be expressed by the product :

$R^1{}_0 = R_{y,\frac{\pi}{2}} R_{x,\frac{\pi}{4}} R_{z,\frac{\pi}{2}}$

which is equivalent to the product of these rotation matrices :

$$\begin{pmatrix} \cos\frac{\pi}{2} & 0 & \sin\frac{\pi}{2} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{2} & 0 & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The above product is evaluated to be the following matrix :

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

Question 2.8

The total result can be expressed by the product :

$R^2{}_0 = R_{y,\frac{\pi}{2}} R_{x,\frac{\pi}{2}}$

which is equivalent to the product of these rotation matrices :

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

The above product is evaluated to be the following matrix :

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Question 2.9

The resulting matrix $\mathbf{R}^3{}_2$ can be expressed by the product :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The above product is evaluated to be the following matrix :

$$\begin{pmatrix} 0 & 0 & -1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

Question 2.12

The matrix $\mathbf{R}_{k,\theta}$ is evaluated to be the following matrix :

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} \\ \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} \\ \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}$$

Question 2.14

The transformation matrix $\mathbf{R}$ can be expressed by the product :

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The above product is evaluated to be the following matrix :

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

We can get K by multiplying the matrix of the differences of the diagonals of the above result by $\frac{1}{2}\,\mathbf{sin}\,\theta$, which gives the result :

$$\begin{pmatrix} 0.357 \\ 0.863 \\ 0.357 \end{pmatrix}$$

Question 2.15

The transformation matrix $R^0{}_1$ can be expressed by :

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Having an x direction of $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$

Question 2.16

The transformation matrix T can be expressed by the product :

$$T = T_{y,1}\ T_{x,3-z,\frac{\pi}{2}}$$

which is equivalent to the product of these rotation matrices :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The above product is evaluated to be the following matrix :

$$\begin{pmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Question 2.17

The transformation matrix $T^2{}_0$ can be expressed by the product :

$$T^2{}_0 = T^1{}_0\, T^2{}_0$$

which is equivalent to the product of these matrices :

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The above product is evaluated to be the following matrix :

$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The above matrix is identical with the matrix $\mathbf{T}^2{}_1$

Question 2.19

The transformation matrix $\mathbf{T}^2{}_0$ can be expressed by the product :

$\mathbf{T}^2{}_0 = \mathbf{T}^1{}_0\,\mathbf{T}^2{}_1$

which is equivalent to the product of these matrices :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{10} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The above product is evaluated to be the following matrix :

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{11}{10} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
