CHAPTER 2

2–1 Let $R = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]$ where $\mathbf{r}_i = \begin{pmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{pmatrix}$. Then $R^T R = I$ implies

$$\begin{bmatrix} \mathbf{r}_1^T\mathbf{r}_1 & \mathbf{r}_1^T\mathbf{r}_2 & \mathbf{r}_1^T\mathbf{r}_3 \\ \mathbf{r}_1^T\mathbf{r}_1 & \mathbf{r}_2^T\mathbf{r}_2 & \mathbf{r}_2^T\mathbf{r}_3 \\ \mathbf{r}_3^T\mathbf{r}_1 & \mathbf{r}_2^T\mathbf{r}_2 & \mathbf{r}_3^T\mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Equating entries of the matrices shows that the column vectors of R are of unit length and mutually orthogonal.

2–2 For any matrices A and B, $det(A^T) = det(A)$ and $det(AB) = det(A)det(B)$. Thus if R is orthogonal

$$1 = det(I) = det(R^T R) = det(R^T)det(R) = det(R)^2$$

which implies that

$$det\ R = \pm 1.$$

2–3 For a right-handed coordinate system, $\mathbf{r}_1 \times \mathbf{r}_2 = \mathbf{r}_3$. This implies that

$$r_{12}r_{23} - r_{13}r_{22} = r_{31}\ ; \quad -r_{11}r_{23} + r_{13}r_{21} = r_{32}\ ; \quad r_{11}r_{22} - r_{12}r_{21} = r_{33}.$$

Therefore, expanding $det\ R$ about column 3 gives

$$\begin{aligned} det\ R &= det \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \\ &= r_{31}(r_{12}r_{23} - r_{22}r_{13}) - r_{32}(r_{11}r_{23} - r_{21}r_{13}) + r_{33}(r_{11}r_{22} - r_{21}r_{12}) \\ &= r_{31}(r_{31}) + r_{32}(r_{32}) + r_{33}(r_{33}) \\ &= \|\mathbf{r}_3\|^2 = 1 \end{aligned}$$

2–4 Equation (2.1.14) is obvious. Equation (2.1.5) follows from

$$R_{z,\theta}R_{z,\phi} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\phi - s_\theta s_\phi & -c_\theta s_\phi - c_\phi s_\theta & 0 \\ s_\theta c_\phi + c_\theta s_\phi & -s_\theta s_\phi + c_\theta c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos(\theta+\phi) & -sin(\theta+\phi) & 0 \\ sin(\theta+\phi) & cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\ \theta+\phi}$$

Equation (2.1.16) follows from (2.1.14) and (2.1.15) since

$R_{z,\theta}R_{z,-\theta} = R_{z,\theta-\theta} = R_{z,0} = I$.

This can also be shown by noticing that

$R^T_{z,\theta} = R_{z,-\theta}$.

2–5 For a rotation of θ about the x axis we have

$i_0 \cdot i_1 = 1$
$j_0 \cdot j_1 = cos\theta$
$k_0 \cdot k_1 = cos\theta$
$k_0 \cdot j_1 = sin\theta$
$j_0 \cdot k_1 = -sin\theta$

and all other dot products are zero.

Substituting into the transformation matrix (2.1.7) gives

$$R^1_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}.$$

For a rotation of θ about the y axis we have

$j_0 \cdot j_1 = 1$
$i_0 \cdot i_1 = cos\theta$
$k_0 \cdot k_i = cos\theta$
$k_0 \cdot i_1 = -sin\theta$
$i_0 \cdot k_1 = sin\theta$

and all other dot products are zero.

Again using (2.1.7) gives

$$R = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

2–6 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(3)$.

From Cramer's rule and the fact that $A \in SO(3)$ we have

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

which implies that $a = d$ and $b = -c$.

Thus $A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$ with $detA = 1 = a^2 + c^2$.

Define $\theta = tan^{-1}(c/a)$. Then $cos\theta = a$ and $sin\theta = c$.

2–7

$$R_0^1 = R_{y,\ \pi/2}\ \ R_{x,\ \pi/4}\ \ R_{z,\ \pi/2}$$

$$= \begin{bmatrix} c_{\pi/2} & 0 & s_{\pi/2} \\ 0 & 1 & 0 \\ -s_{\pi/2} & 0 & c_{\pi/2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{\pi/4} & -s_{\pi/4} \\ 0 & s_{\pi/4} & c_{\pi/4} \end{bmatrix} \begin{bmatrix} c_{\pi/2} & -s_{\pi/2} & 0 \\ s_{\pi/2} & c_{\pi/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

2–8

$$R_0^2 = R_{y,\ \pi/2} R_{x,\ \pi/2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

2–9

$$R_2^3 = R_2^1\, R_1^3 \quad \text{where} \quad R_2^1 = (R_1^2)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Therefore

$$R_2^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \end{bmatrix}$$

2–11 If λ is an eigenvalue of R and $\mathbf{k}$ is a unit eigenvector corresponding to λ then, $R\mathbf{k} = \lambda\mathbf{k}$. Since R is a rotation $\|R\mathbf{k}\| = \|\mathbf{k}\|$. This implies that $|\lambda| = 1$, i.e., the eigenvalues of R are on the unit circle in the complex plane. Since the characteristic polynomial of R is of degree three at least one eigenvalue of R must be real. Hence $+1$ or -1 is an eigenvalue of R. Now, since $+1 = det R = \lambda_1\lambda_2\lambda_3$ where $\{\lambda_1,\ \lambda_2,\ \lambda_3\}$ is the set of eigenvalues of R, it is easy to see that if -1 is an eigenvalue then $\{\lambda_1,\ \lambda_2,\ \lambda_3\} = \{-1,\ -1,\ +1\}$. In any case $+1$ is always an eigenvalue of R.

The vector $\mathbf{k}$ defines the axis of rotation in the angle/axis representation of R.

2–12

$$R_{\mathbf{k},\ \theta} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} \\ \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} \\ \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} \end{bmatrix}$$

2–14

$$R = R_{y,90}\, R_{z,\ 45} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$\theta = cos^{-1}\left(\frac{TR(R) - 1}{2}\right) = cos^{1}\left(\frac{\frac{\sqrt{2}}{2} - 1}{2}\right) = 98.42°$$

$$K = \frac{1}{2\ sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = (0.5054481) \begin{bmatrix} 0.7071068 \\ 1.7071068 \\ 0.7071068 \end{bmatrix}$$

2–15

$$R_0^1 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$ The direction of the x-axis is

$$\mathbf{i}_1 = (0,\ \tfrac{1}{\sqrt{2}},\ -\tfrac{1}{\sqrt{2}})^T$$

2–16

$$T = T_{y,1}T_{x,3}\overset{R}{\cancel{T}}_{z,\pi/2}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2–17

$$H_0^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_0^2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_1^2 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2–18

$$H_0^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_0^2 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_0^3 = \begin{bmatrix} 0 & 1 & 0 & -.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2–19

$$H_1^2 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & .5 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^2 = H_0^1\, H_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & .5 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^3 = H_0^2\, H_2^3 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & -1 & 0 & 1.5 \\ 0 & 0 & -1 & 3.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2–20

$$H_2^3 = \begin{bmatrix} 1 & 0 & 0 & -.3 \\ 0 & -1 & 0 & .4 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The homogeneous transformation from the block frame to the base frame is

$$H_0^2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & .8 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2–21

$$S(\mathbf{a})\mathbf{p} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -a_z p_y + a_y p_z \\ a_z p_x - a_x p_z \\ -a_y p_x + a_x p_y \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{p} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ p_x & p_y & p_z \end{bmatrix} = \mathrm{i}(a_y p_z - a_z p_y) - \mathrm{j}(a_x p_z - a_z p_x) + \mathbf{k}(a_x p_y - a_y p_x).$$

Therefore $S(\mathbf{a})\mathbf{p} = \mathbf{a} \times \mathbf{p}$

2–23

$$R_{x,\,90} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; \qquad S(\mathbf{a}) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}; \qquad R\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$S(R\mathbf{a}) = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Then

$$RS(\mathbf{a})R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = S(R\mathbf{a}).$$

2–24

$R_0^1 = T_{x,\theta} R_{y,\phi}$. Then

$$\frac{\partial R_0^1}{\partial \phi} = R_{x,\,\theta}\frac{\partial R_{y,\,\phi}}{\partial \phi} = R_{x,\,\theta} S(j) R_{y,\,\phi} = \begin{bmatrix} -s\phi & 0 & c\phi \\ s\theta c\phi & 0 & s\theta S\phi \\ -c\theta c\phi & 0 & -s\phi c\theta \end{bmatrix}$$

$$\frac{\partial R_0^1}{\partial \phi}\Big|_{\substack{\theta=\pi/2 \\ \phi=\pi/2}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

2–25 By direct calculation

$$I + S(\mathbf{k})s_\theta + S^2(\mathbf{k})v_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -k_z s_\theta & k_y s_\theta \\ k_z s_\theta & 0 & -k_x s_\theta \\ -k_y s_\theta & k_x s_\theta & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} (-k_z^2 - k_y^2)v_\theta & k_x k_y v_\theta & k_x k_z v_\theta \\ k_x k_y v_\theta & (-k_z^2 - k_x^2)v_\theta & k_y k_z v_\theta \\ k_x k_z v_\theta & k_y k_z v_\theta & (-k_y^2 - k_x^2)v_\theta \end{bmatrix}$$

Adding the three matrices and using $k_x^2 + k_y^2 + k_z^2 = 1$ yields (2.2.16).

2–26

$$dR_{y,\theta}/d\theta \; R_{y,\theta}^T = \begin{bmatrix} -s\theta & 0 & c\theta \\ 0 & 0 & 0 \\ -c\theta & 0 & -s\theta \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = S(\mathrm{j})$$

$$dR_{z,\theta}/d\theta \; R_{z,\theta}^T = \begin{bmatrix} -s\theta & -c\theta & 0 \\ c\theta & -s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S(\mathbf{k})$$

2–27 $S(\mathbf{k})^3 = -S(\mathbf{k})$ can be verified by direct multiplication.

To show (2.5.20) we compute using Problem 2-25

$$\frac{dR}{d\theta} = S(\mathbf{k})\;cos\;\theta + S^2(\mathbf{k})\;sin\;\theta$$

also from Problem 2-25

$$\begin{aligned} S(\mathbf{k})R_{\mathbf{k},\theta} &= S(\mathbf{k}) + S^2(\mathbf{k})\;sin\theta + S^3(\mathbf{k})(1 - \;cos\theta) \\ &= S(\mathbf{k})\;cos\theta + S^2(\mathbf{k})\;sin\theta \end{aligned}$$

Using the fact that $S^3(\mathbf{k}) = -S(\mathbf{k})$.

2–28 If $S \in SS(3)$ then

$$(e^S)^T = e^{S^T} = e^{-S}$$

which can be verified using the series definition for e^S. Therefore

$$e^S \cdot (e^S)^T = e^S e^{-S} = e^{S-S} = e^0 = I$$

Also

$$det(e^S) = e^{Tr(S)} = e^0 = 1$$

Hence $e^S \in SO(3)$.

2–29

$$\begin{aligned}
e^{S(\mathbf{k})\theta} &= I + S\theta + \frac{\theta^2}{2!}S^2 + \frac{\theta^3}{3!}S^3 + \ldots \\
&= I + S\theta + \frac{\theta^2}{2!}S^2 + \frac{\theta^3}{4!}(-S^2) + \ldots \\
&= I + S(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots) + S^2(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots) \\
&= I + S(\mathbf{k})\ sin\ \theta + S^2(\mathbf{k})(1 - cos\theta) \\
&= I + S(\mathbf{k})\ sin\theta + S^2(\mathbf{k})(vers(\theta)) = R_{\mathbf{k},\theta}
\end{aligned}$$

2–30 From 2–27 we have

$$\frac{dR}{d\theta} = S(\mathbf{k})R(\theta)\,.$$

The solution of this differential equation is

$$R(\theta) = R(0)e^{S(\mathbf{k})\theta} = e^{S(\mathbf{k})\theta}.$$

2–32

$$\dot{R}_0^1 = \dot{R}_{z,\ \phi}R_{y,\ \theta}R_{x,\ \psi} + R_{z,\ \phi}\dot{R}_{y,\ \theta}R_{x,\ \psi} + R_{z,\ \phi}R_{y,\ \theta}\dot{R}_{x,\ \psi}$$

The first term above is just

$$\dot{\phi}S(\mathbf{k})R_{z,\ \phi}R_{y,\ \theta}R_{x,\ \psi} = \dot{\phi}S(\mathbf{k})R_0^1$$

The second term is

$$\begin{aligned}
R_{z,\phi}\dot{\theta}S(\mathrm{j})R_{y,\ \theta}R_{x,\ \psi} &= \dot{\theta}R_{z,\ \phi}S(\mathrm{j})R_{z,\ \phi}^T R_{z,\ \phi}R_{y,\ \theta}R_{x,\ \psi} \\
&= \dot{\theta}S(R_{z,\ \phi}\mathrm{j})R_0^1
\end{aligned}$$

Similarly, the third term is

$$R_{z,\ \phi}R_{y,\ \theta}\dot{\psi}S(\mathrm{i})R_{x,\ \psi} = \dot{\psi}S(R_{z,\ \phi}R_{y,\ \theta}\mathrm{i})R_0^1$$

Adding the three terms and using linearity of $SS(3)$ gives

$$\omega_0^1 = \dot{\phi}\mathbf{k} + \dot{\theta}R_{z,\,\phi}\mathbf{j} + \dot{\psi}R_{z,\,\phi}R_{y,\,\theta}\mathbf{i}$$

$$= \begin{bmatrix} c_\phi c_\theta \dot{\psi} - S_\phi \dot{\theta} \\ S_\phi c_\theta \dot{\psi} + c_\phi \dot{\theta} \\ \dot{\phi} - S_\theta \dot{\phi} \end{bmatrix}$$

2–33

$$p_0 = Rp_1 + d$$
$$\dot{p}_0 = R\dot{p}_1$$
$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

2–34

$$\omega_0^2 = \omega_0^1 + R_0^1\omega_1^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$