PROBLEMS

2-1 If R is an orthogonal matrix show that the column vectors of R are of unit length and mutually perpendicular.

2-2 If R is an orthogonal matrix show that $\det R = \pm 1$.

2-3 Show that $\det R = +1$ if we restrict ourselves to right-handed coordinate systems.

2-4 Verify Equations 2.1.14–2.1.16.

2-5 Derive Equations 2.1.17 and 2.1.18.

2-6 Suppose A is a 2×2 rotation matrix. In other words $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2-7 Find the rotation matrix representing a roll of $\frac{\pi}{4}$ followed by a yaw of $\frac{\pi}{2}$ followed by a pitch of $\frac{\pi}{2}$.

2-8 If the coordinate frame $o_1x_1y_1z_1$ is obtained from the coordinate frame $o_0x_0y_0z_0$ by a rotation of $\frac{\pi}{2}$ about the x-axis followed by a rotation of $\frac{\pi}{2}$ about the fixed y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

2-9 Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$, and $o_3x_3y_3z_3$ are given, and suppose

$$R_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}; \quad R_1^3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the matrix R_2^3.

2-10 Verify Equation 2.2.16.

2-11 If R is a rotation matrix show that +1 is an eigenvalue of R. Let $\mathbf{k}$ be a unit eigenvector corresponding to the eigenvalue +1. Give a physical interpretation of $\mathbf{k}$.

2-12 Let $\mathbf{k} = \frac{1}{\sqrt{3}}(1,1,1)^T$, $\theta = 90°$. Find $R_{\mathbf{k},\theta}$.

2-13 Show by direct calculation that $R_{\mathbf{k},\theta}$ given by (2.2.16) is equal to R given by (2.3.5) if θ and $\mathbf{k}$ are given by (2.3.6) and (2.3.7), respectively.

2-14 Suppose R represents a rotation of 90° about y_0 followed by a rotation of 45° about z_1. Find the equivalent axis/angle to represent R. Sketch the initial and final frames and the equivalent axis vector $\mathbf{k}$.

2-15 Find the rotation matrix corresponding to the set of Euler angles $(\frac{\pi}{2}, 0, \frac{\pi}{4})$. What is the direction of the x_1 axis relative to the base frame?

2-16 Compute the homogeneous transformation representing a translation of 3 units along the x-axis followed by a rotation of $\frac{\pi}{2}$ about the current z-axis followed by a translation of 1 unit along the fixed y-axis. Sketch the frame. What are the coordinates of the origin o_1 with respect to the original frame in each case?

2-17 Consider the diagram of Figure 2-10. Find the homogeneous transformations H_0^1, H_0^2, H_1^2 representing the transformations among the three frames shown. Show that $H_0^2 = H_0^1 H_1^2$.

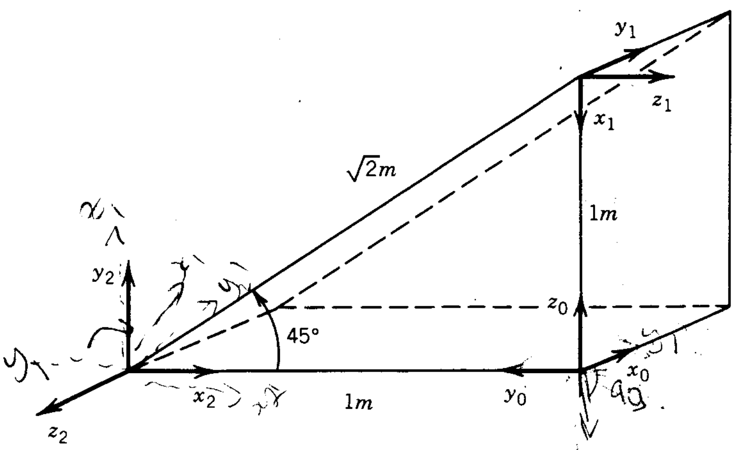

FIGURE 2-10
Diagram for Problem 2-17.

2-18 Consider the diagram of Figure 2-11. A robot is set up 1 meter from a table, two of whose legs are on the y_0 axis as shown. The table top is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2 m above the table top with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.

2-19 In Problem 2-18, suppose that, after the camera is calibrated, it is rotated 90^o about the axis z_3. Recompute the above coordinate transformations.

2-20 If the block on the table is rotated 90^o degrees about the axis z_2 and moved so that its center has coordinates $(0,.8,.1)^T$ relative to the frame $o_1x_1y_1z_1$, compute the homogeneous transformation relating the block frame to the camera frame; the block frame to the base frame.

2-21 Verify Equation 2.5.7 by direct calculation.

2-22 Prove assertion (2.5.8) that $R(\mathbf{a}\times\mathbf{b}) = R\mathbf{a}\times R\mathbf{b}$, for $R \in SO(3)$.

2-23 Suppose that $\mathbf{a} = (1\ ,-1\ ,2)^T$ and that $R = R_{x,90}$. Show by direct calculation that

$$RS(\mathbf{a})R^T = S(R\,\mathbf{a})$$

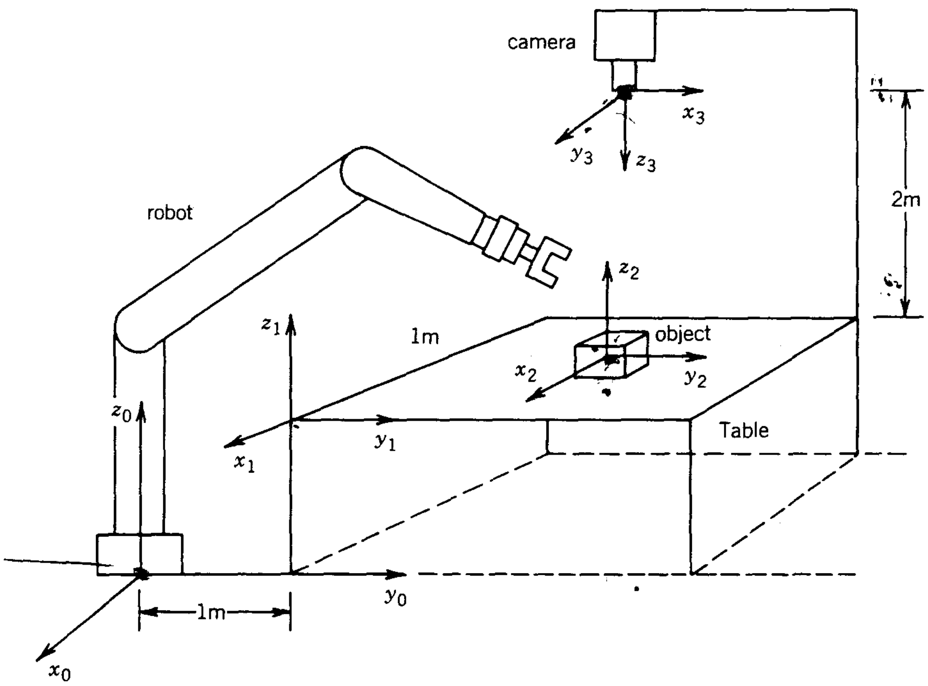

FIGURE 2-11

Diagram for Problem 2-18.

2-24 Given $R_0^1 = R_{x,\theta}R_{y,\phi}$, compute $\frac{\partial R_0^1}{\partial \phi}$. Evaluate $\frac{\partial R_0^1}{\partial \phi}$ at $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$.

2-25 Use Equation 2.2.16 to show that

$$R_{\mathbf{k},\theta} = I + S(\mathbf{k})\sin(\theta) + S^2(\mathbf{k})\text{vers}(\theta)$$

2-26 Verify (2.5.19) by direct calculation.

2-27 Show that $S(\mathbf{k})^3 = -S(\mathbf{k})$. Use this and Problem 2-25 to verify Equation 2.5.20.

2-28 Given any square matrix A, the exponential of A is a matrix defined as

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$$

Given $S \in SS(3)$ show that $e^S \in SO(3)$.
[Hint: Verify the facts that $e^A e^B = e^{A+B}$ provided that A and B commute, that is, $AB = BA$, and also that $\det(e^A) = e^{Tr(A)}$.]

2-29 Show that $R_{\mathbf{k},\theta} = e^{S(\mathbf{k})\theta}$.
[Hint: Use the series expansion for the matrix exponential together with Problems 2-25 and 2-27. Alternatively use the fact that $R_{\mathbf{k},\theta}$ satisfies the differential equation

$$\frac{dR}{d\theta} = S(\mathbf{k})R$$

2-30 Use Problem 2-29 to show the converse of 2-28, that is, if $R \in SO(3)$ then there exists $S \in SS(3)$ such that $R = e^{S}$.

2-31 Given the Euler angle transformation

$$R = R_{z,\psi}R_{y,\theta}R_{z,\phi}$$

show that $\frac{d}{dt}R = S(\boldsymbol{\omega})R$ where

$$\boldsymbol{\omega} = (c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta})\mathbf{i} + (s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta})\mathbf{j} + (\dot{\psi} + c_\theta \dot{\phi})\mathbf{k}.$$

The components of **i**,**j**,**k**, respectively, are called the **nutation, spin, and precession.**

2-32 Repeat Problem 2-31 for the Roll-Pitch-Yaw transformation. In other words, find an explicit expression for $\boldsymbol{\omega}$ such that $\frac{d}{dt}R = S(\boldsymbol{\omega})R$, where R is given by (2.3.10).

2-33 Two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the homogeneous transformation

$$H = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A particle has velocity $\mathbf{v}_1(t) = (3,1,0)^T$ relative to frame $o_1x_1y_1z_1$. What is the velocity of the particle in frame $o_0x_0y_0z_0$?

2-34 Three frames $o_0x_0y_0z_0$, $o_1x_1y_1z_1$, and $o_2x_2y_2z_2$ are given below. If the angular velocities $\boldsymbol{\omega}_0^1$ and $\boldsymbol{\omega}_1^2$ are given as

$$\boldsymbol{\omega}_0^1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \boldsymbol{\omega}_1^2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

what is the angular velocity $\boldsymbol{\omega}_0^2$ at the instant when

$$R_0^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$